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DESIGN OF FRAMES BY RELAXATION OF YIELD-HINGES

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STRUCTURAL DIVISION

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PAPERS

DESIGN OF FRAMES BY RELAXATION OF YIELD-HINGES

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SYNOPSIS

A method is herein developed for determining the loading condition that causes the collapse of a rigid frame structure. It is shown that the location of the "yield-hinges," when the structure is on the verge of collapse, is independent of the stiffness of the individual members. The method consists of (1) the arbitrary distribution of moment to the joints of the structure, and (2) the adjustment of this distribution by the trial-and-error method until the highest ratio of actual moment to limit moment—at any point in the structure—reaches a minimum. The location of the yield-hinges are the points at which these minima occur.

The procedure is extended to account for the order in which the yield-hinges develop. This order is dependent on the stiffnesses of the members and, therefore, the initial moment distribution must be consistent with the actual elastic properties of the structure. Similarly, the subsequent redistribution of moment—to minimize the highest values—is made in accordance with the elastic properties of the structure.

INTRODUCTION

If a system of loads acting on a rigid frame structure is increased slowly and proportionately, some point within the structure will reach its limiting elastic value of resisting moment. As the loading increases, the resisting moment at this point will increase at a diminishing rate, and the moment will increase slowly as the values of moment at other locations will tend to approach it. As other moments approach the end of the elastic range, corresponding points will in turn offer progressively less resistance to bending until a sufficient number of points attain their limiting moment values. When this occurs, the structure, as a part or as a whole, cannot effectively offer any further resistance

to the loading and collapse becomes imminent. The magnitude of the loads at collapse, or at an arbitrarily defined deformation of the structure, should be considered as the true criterion for design rather than the surpassing of the elastic limit of the material at any point in the structure.

It is usually postulated that the relation between moment and angle change is as depicted in Fig. 1(*c*) rather than as shown in Fig. 1(*b*). Fig. 1(*b*) is derived from an assumption of the flat-topped stress-strain relationship shown in Fig. 1(*a*). The moment is considered to reach a maximum value M' without any angle change. After attaining this value, the moment is considered to remain constant for an unlimited angle change. Thus a point on a structure is a yield-hinge when the moment at that point is equal to M'.

Because the deformation in the elastic range and the elastic-plastic range is relatively insignificant in comparison to the subsequent unrestricted deformation, the usual postulate is unnecessarily restrictive. The loading path is unimportant if a constant maximum moment can be attained. The yield-hinge can develop as a result of any actual stress-strain relation or because of instability of the flanges or web of a flanged section. However, the actual deflection of the structure will depend on the relationship between moment and angle change. Where it is required to determine this deflection, the assumption of a flat-topped relationship might not be correct.

Conditions of Collapse

H. J. Greenberg and W. Prager, M. ASCE, have demonstrated a solution for the problem of the rigid frame structure wherein a number, λ, by which the actual loads must be multiplied to bring about collapse, is determined by guessing the collapse mode (that is, the location of the yield-hinges to produce collapse), and then verifying the validity of the guess by checking the statics and the yield conditions.[2] If the guess should prove to be incorrect, a yield condition will be found to be violated at some point. A revised guess is then made and the process is repeated until the correct solution is found. The principal disadvantage of this method is the inability of the analyst to guess the correct mode at the outset of the solution.

[2] "Limit Design of Beams and Frames," by H. J. Greenberg and W. Prager, *Transactions*, ASCE, Vol. 117, 1952, p. 447.

If it were known that the correct location of the yield-hinges is as shown in Fig. 2, the solution would be simple. In effect, the structure has been reduced to a mechanism by the introduction of the yield-hinges. The structure cannot support any further increase of load and so will collapse.

When the structure deflects a distance δ, measured horizontally at the point of application of the load, the external work is $\lambda P \delta$. The energy stored in the frame at any joint i will be $M'_i \theta_i$ (Fig. 1(*c*)). The assumption of no previous elastic deformation excludes consideration of elastic energy. Equating the internal work to the external work,

$$\sum \lambda P \delta = \sum M'_i \theta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which M'_i is the limit moment at joint i and θ_i is the angular deformation at joint i.

The angle change θ can always be related, by the geometry of the structure,

to the deflection δ; thus,

$$\lambda = \frac{11\ M'}{3\ P\ h} \qquad (2)$$

Determination of Yield-Hinges.—The solution proposed by Messrs. Greenberg and Prager is easily obtained, provided the location of the yield-hinges can be predicted. The trial-and-error approach may be suitable for uncomplicated frames but becomes too complex for intricate structures. The method to be developed subsequently overcomes this difficulty by establishing the location of the yield-hinges as part of the solution. Another method which overcomes the difficulty inherent in the method presented by Messrs. Greenberg and Prager has been developed by P. S. Symonds and B. G. Neal.[3]

[3] "Recent Progress in the Plastic Methods of Structural Analysis," Pt. I by P. S. Symonds and B. G. Neal, *Journal*, The Franklin Institute, November, 1951, pp. 383–407; Pt. II, December, 1951, pp. 469–491.

It is significant to note that nowhere in the development of Eq. 1 was it necessary to introduce the concept of member stiffness, whereas the elastic solution is dependent on the stiffness. The maximum value of M/M' in the elastic solution of the frame will determine the location of the first yield-hinge. The location of the maximum value of M/M' is dependent on the relative stiffnesses. Likewise, the order in which the other hinges develop is dependent on the member stiffnesses. The following principle may therefore be stated: The order in which yield-hinges develop, as all the loads on the structure are increased proportionally, will depend on the relative stiffnesses of the members of the structure. However, irrespective of the order in which the yield-hinges develop, the final locations of the yield-hinges will be unaltered.

This principle permits the arbitrary assignment of any value of member stiffness provided only that the final collapse condition is sought. Any arbitrary distribution of moment to the various joints of the frame—consistent with the requirements of statics—is tantamount to an arbitrary assignment of stiffnesses. Therefore, it is possible to obtain a moment distribution such that the maximum value of M/M' is the same at all points at which yield-hinges will develop. As the loads are increased, all such points will reach their limit moments simultaneously and all yield-hinges will form concurrently. Hence, the problem reduces to one of imagining an elastic structure designed to satisfy this requirement.

It is evident that in such a case the ultimate load will be greater than the load found from an elastic analysis by the ratio of the ultimate moment at the yield-hinges to the moment at the elastic limit. As can readily be shown, this increase may be negligible for flanged members. Under such conditions, the elastic solution and the limit solution are approximately identical.

Although the questions of shakedown and of limiting deflection are not to be dealt with herein, it is of interest to note that an elastic structure which is designed to have all yield-hinges form simultaneously is an optimum design on the basis of both of these questions.

It can be noted that the moment distribution which produces the same value for the maximum value of M/M' at all points that are to be yield-hinges will be that for which the maximum value of M/M' in the structure is minimized. This is implicit in Theorem I of Messrs. Greenberg and Prager[2] and it can be readily seen with reference to a single-span restrained beam.

The moment diagram CDEF in Fig. 3 is a primary statics requirement for a simple-beam loading condition. The effect of end restraints superimposes the reference axis AB. Collapse will occur when the value of M/M' reaches unity at points 1, 2, and 3. Since it is required that the ratio M/M' for all three points reach unity simultaneously, the axis line AB is fixed such that

$$\frac{M_1}{M'_1} = \frac{M_2}{M'_2} = \frac{M_3}{M'_3} = \text{a minimum} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

F. Panlillio has applied this principle to a graphic solution of the problem of continuous beams.[4]

[4] "Theory of Limit Design Applied to Magnesium Alloy and Aluminum Alloy Structures," by F. Panlillio, *Journal*, Royal Aeronautical Soc., June, 1947.

Since it would be difficult to determine analytically the moment distribution which would satisfy the requirements of Eqs. 3, any arbitrary distribution can be assumed. The point at which the maximum ratio of M_i/M'_i occurs will be a yield-hinge.

The loads can be considered to be large enough to cause the arbitrary elastic solution to have at least one moment in excess of the limit moment. Because a value of moment in excess of M' is a violation of the yield condition, the excess moment at the point must be redistributed (or relaxed) to locations that can accept the increased moment. The mechanics of this relaxation process are best illustrated by the following solutions of problems.

Example 1.—The frame shown in Fig. 4 is loaded as indicated in Fig. 4(*a*). For simplicity, the limit moment is assumed to be the same for all points; that is M' is equal to 100. The problem requires the determination of a multiplier λ by which the loads must be increased in order to cause collapse of the frame. Since M' is constant throughout, the moment values, rather than the ratios, are considered.

Conditions of statics require that, regardless of the actual distribution of moments at the ends of the members, the member BC must support the simple beam moment of $\frac{P\,L}{4}$ and that the vertical members AB and CD must support the lateral moment $H\,h$. The distribution of moment might be guessed as $\frac{H\,h}{4} = 20$, acting at each end of each vertical member. This would leave $\frac{P\,L}{4} = 25$ to be resisted under load P. The moment values are recorded in Fig. 4(*b*). It can be seen that such a distribution of moment would correspond to an infinite stiffness of the member BC.

The maximum moment value of 25 at P indicates that this point will be a yield-hinge when P is increased by a factor λ. The value of 25 can be considered to have already exceeded the limit value and so must be reduced. That is, the yield-hinge at P must be relaxed by distributing the excess moment to some other point or points.

The distribution of moment which revealed that a yield-hinge will be formed at P was arbitrary. Any other distribution that would be statically correct could also have been assumed, although it probably would have revealed a different location for the first yield-hinge. For example, an assumption that the horizontal force was resisted entirely by equal moments at points A and D would indicate that these points would be yield-hinges. A series of such as-

sumptions can thus reveal very quickly the locations of most of the yield-hinges. This technique may also indicate the manner of blocking relaxations for the determination of the remaining yield-hinges. If the distribution first assumed is used, the procedure can be continued as follows:

1. A value by which the moment M_P under the load P can be reduced is estimated as 4 in this case. Then M_P will equal 21.

2. Because the object is to make a number of points reach the same final values, it is assumed that the change in M_P will produce a change in some other moment (or moments) to make it equal to a value of 21. Any moment can be used, but in this case the moment M_{CB} (at point C in member BC) is chosen. This nomenclature for representing moments will be used throughout this paper. That is, the moment will act at the end of the member corresponding to the first subscript and the moment will occur in the member whose ends are denoted by the two subscripts.

3. The moment M_{BC} is adjusted to satisfy the requirements for statical equilibrium of member BC. This requires that the moment M_{BC} equal 13.

4. The moments in the vertical members are adjusted to satisfy the requirements for statical equilibrium. To do this the moment M_{AB} is made equal to the moment M_{DC}. Both of these moments are equal to 23. This value, since it is greater than 21, indicates that the estimated reduction of M_P by 4 was too large. However, it is apparent that the final yield-hinges are located at points A, C, and D and under the load P. The change which was effected caused a reduction in the moment M_{BC}. Likewise, for any other change it would be impossible to make the value of M_{BC} exceed that of the other points that were shown to be yield-hinges.

5. The process is repeated by relaxing the excess values at points A and D or by revising the reduction of M_P by 4, until all four points have equal values. In this problem, this occurs at $M_{AB} = M_{CB} = M_{DC} = M_P = 21.7$.

6. The multiplier λ can be found by dividing M' by the final relaxed value of the maximum moment in the frame; thus $\lambda = \dfrac{100}{21.7} = 4.61$.

Example 2.—The frame of Fig. 2 is analyzed in Fig. 5. The sign convention adopted assumes that the moment at the end of a member is positive in the clockwise sense.

As in Example 1, it is not important what the initial distribution of moment is assumed to be. If the location of the yield-hinges were known, or could be guessed, the relaxation process might be completed more readily, as in Example 1. However, the solution is readily determinable and it is not advantageous to spend time attempting to anticipate the location of the hinges.

In Example 2 the assumed distribution corresponds to infinite stiffness in the horizontal members and equal stiffnesses in the vertical members. The moment M_{FJ} is the largest and hence the first yield-hinge. Relaxing the excess moment M_{FJ}, it is seen that the moment M_{BJ} next reaches a maximum. The moments M_{FJ} and M_{BJ} can be relaxed simultaneously until the moment M_{GF} appears as a yield-hinge. The process is continued, reducing by arbitrary increments all hinges that have been determined, until a minimum value, below which it is impossible to go without violating the requirement of statical equilibrium, is obtained. The final values are recognized by: (*a*) the fact that the same value is obtained at a sufficient number of points so that if these

points were hinges, the structure would become a mechanism and (*b*) any further attempted reduction of moment at one or more points will necessitate an increase of moment at some other point. The final maximum moment divided into the limit moment M' results in $\lambda = \frac{1,000}{273} = 3.67$.

General Solution.—In Examples 1 and 2, frames in which the limit moments were the same everywhere were investigated to simplify the problem. In an actual structure it is not likely that the limit moment will be the same for each point. The general problem, in which the limit moment can be any value, can be handled by recording the relative values of the limit moment on the frame, after each relaxation, dividing the relaxed value by this relative limit moment. The procedure may be performed as shown previously until the maximum ratio M/M' is minimized. The problem can be handled most readily by choosing a reference limit moment and relating this reference value to the limit moment at any point by a factor r. Then the moments after each distribution can be divided by its corresponding factor r. The r-values can be recorded on the frame as shown in Fig. 6 which is the solution for Example 3.

Example 3.—The complete solution for the frame shown in Fig. 6(*a*) is indicated in Fig. 6(*b*). In this solution local collapse is indicated by the development of hinges at points C, D, and E. However, hinges also are indicated at points A, B, G, H, J, and L. The fact that the moment M_{FJ} at one end of one vertical can become greater makes collapse in a horizontal mode impossible unless it is stipulated that the vertical load on the upper member of the frame be held constant while only the horizontal loads are increased. It should be noted that M_{FJ} is less than a limit moment because of the choice of distribution of moment. The moment could have been less than the limit moment at some other point or points. In fact, the location of the actual hinges in the lower structure is not revealed by the process.

A further generalization is needed for the consideration of the case of a frame having unequal legs. In Example 3 the height of the vertical members in the frame were the same in any one story. This made it possible, during the distribution process, to maintain the total of the moments constant at the ends of the members. Actually, it is the shear which must be constant across any story section. Thus for member AB,

$$H = \frac{(M_{AB} + M_{BA})_i}{h_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Therefore, in the case of a frame having unequal legs it is necessary to recompute the total of the end moments after each distribution.

Example 4.—The frame shown in Fig. 7(*a*) is solved in Fig. 7(*b*). The shear relation of Eq. 4 can be rearranged into a more convenient form to facilitate the solution of this example. Thus

$$M_{AB} + M_{BA} = 80 - \tfrac{2}{3}\,(M_{CD} + M_{DC}) \quad \ldots\ldots\ldots\ldots (5a)$$

or

$$M_{CD} + M_{DC} = 120 - \tfrac{3}{2}\,(M_{AB} + M_{BA}) \quad \ldots\ldots\ldots\ldots (5b)$$

Order of the Development of Yield-Hinges

It has been shown previously that the individual member stiffnesses have

no effect on the final location of the yield-hinges. Nevertheless, the stiffnesses do affect the order in which the yield-hinges form. Inasmuch as the design can be governed by considerations other than the condition for final collapse of the structure—for example, deflection—it often is desirable to know what yield-hinges have developed at intermediate stages. This can be accomplished by extending the application of the relaxation technique with proper regard to the actual stiffnesses of the members. In this case the excess moment at a point must be distributed by the elastic method developed by Hardy Cross, Hon. M. ASCE. However, the accuracy required is low because a relatively large percentage of error in the adjustment of the amount, which is itself only a fraction of the total moment, represents a correspondingly small error in the final answer.

The elastic solution for Example 1 is shown in Fig. 8(*a*). It is seen that the first yield-hinge will occur at joint D as a result of a maximum moment of 29.9. The elimination of the excess moment at joint D can be accomplished by superimposing an opposite moment at that joint. The distribution of an arbitrarily applied moment of 100 at joint D is shown in Fig. 8(*b*), in which the factors for the distribution of moment at each joint are shown in boxes above the appropriate joint. In this case the side sway unbalanced moment equals 128 and is balanced by moments of −44 at joint A, −39 at joint B, and −45 at joint C. The distributed moment values at the various points of the frame can be assumed to be influence coefficients for a moment change at joint D.

By relaxing the moment at joint D an amount of 2 (Fig. 8(*c*)), it is seen that joint C develops a maximum moment. Consequently a set of coefficients must be determined for joint C, as shown in Fig. 8(*c*). In this case the side sway unbalanced moment equals 180 and is balanced by moments of −56 at joint A, −46 at joint B, and −76 at joint D.

The excess moment at joint C is reduced by 5 (Fig. 8(*e*)), but it is noted that the relaxation of the moment at joint C disturbs the value of the moment at joint D. Therefore, the moment at joint D must be reduced. Choosing a value of 5 for this reduction causes the moment at joints C and D to be equal. The moment of 24. 2 is greater than the moment of 21.7 that was completed in Example 1. This necessitates the relaxing of joints C and D simultaneously by adding the coefficients for joints C and D, as shown in Fig. 8(*d*). In Fig. 8(*d*) the sidesway unbalanced moment equals 230 and is balanced by moments of −132 at joint A and −98 at joint B. The distribution shown in Fig. 8(*d*) results in the final moments shown in Fig. 8(*e*).

Conclusion

The method that has been proposed for determining the final mode of collapse of a frame is simple and readily applicable for the solution of relatively complicated frames. The method does not require that the moment be constant with increasing angular distortion but only that the moment reach a maximum value from which it does not diminish.

The determination of the condition of final collapse is not the entire problem. In many cases the limit design of a rigid frame may be governed by (*a*) limit of deformation, (*b*) limit of stability, or (*c*) the need to guard against an inelastic stress reversal. Under these circumstances the ultimate loading will

be less than the collapse value. Hence, a knowledge of the progress of collapse from yield-hinge to yield-hinge is required. The method determines this order of yield-hinge development in relation to the actual stiffnesses of the members.

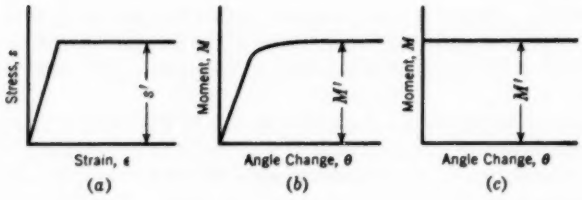


FIG. 1

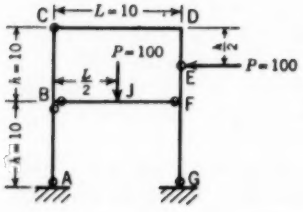


FIG. 2.—RIGID FRAME STRUCTURE

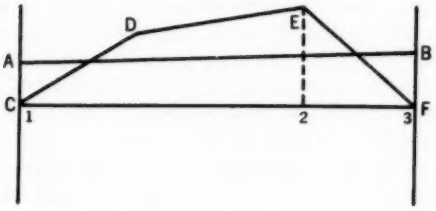


FIG. 3.—MOMENT DIAGRAM

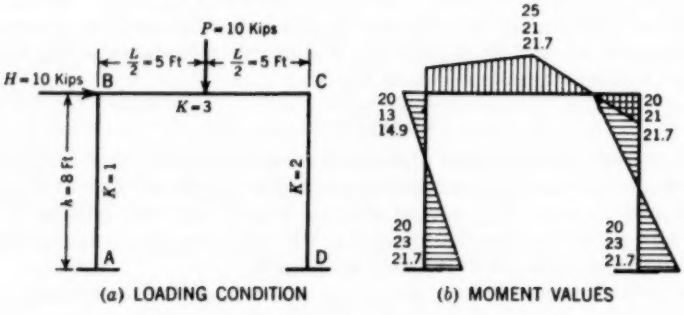


FIG. 4.—SOLUTION OF EXAMPLE 1

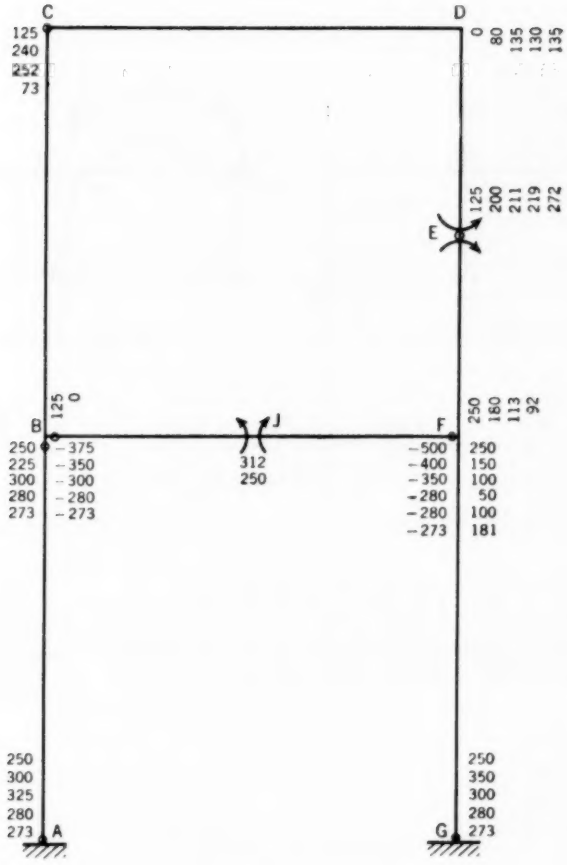
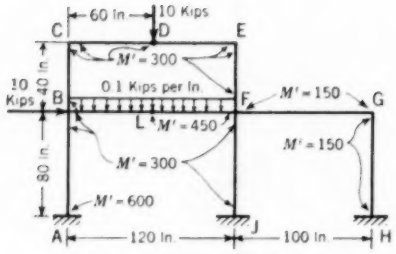
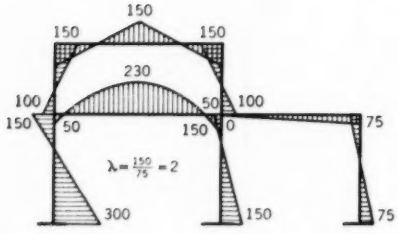


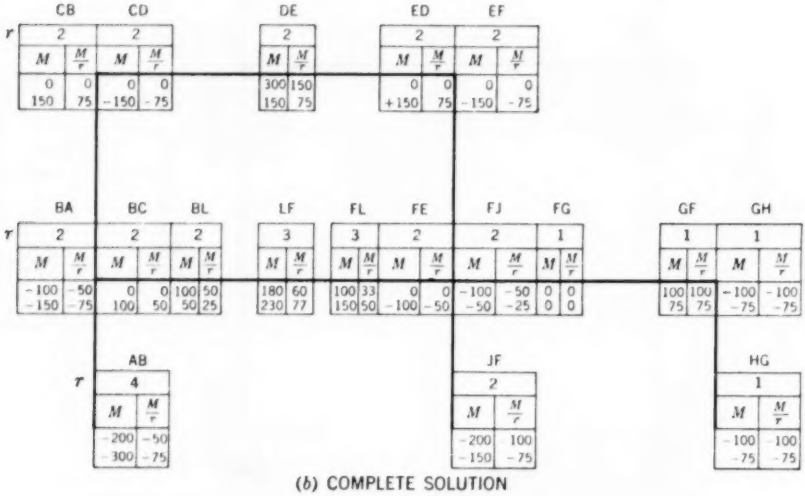
FIG. 5.—SOLUTION OF EXAMPLE 2



(a) LOADING CONDITION

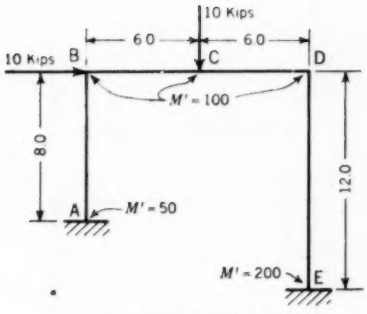


(c) MOMENT VALUES



(b) COMPLETE SOLUTION

FIG. 6.—SOLUTION OF EXAMPLE 3



(a) LOADING CONDITION

	BA		BC		CD		DC		DE	
r	2		2		2		2		2	
	M	M/r	M	M/r	M	M/r	M	M/r	M	M/r
	−30	−15	30	15	38	19	+15	+8	−15	−8
	−50	−25			50	25			−10	−5
	−40	−20			40	20			−20	−10
	−30	−15			30	15			−30	−15

	AB				ED	
r	1				4	
	M	M/r			M	M/r
	−30	−30			−15	−4
	−25	−25			+2.5	
	−20	−20			−10	−2
	−15	−15			−23	−6

$\lambda = \frac{50}{15} = 3.33$

(b) COMPLETE SOLUTION

FIG. 7.—SOLUTION OF EXAMPLE 4

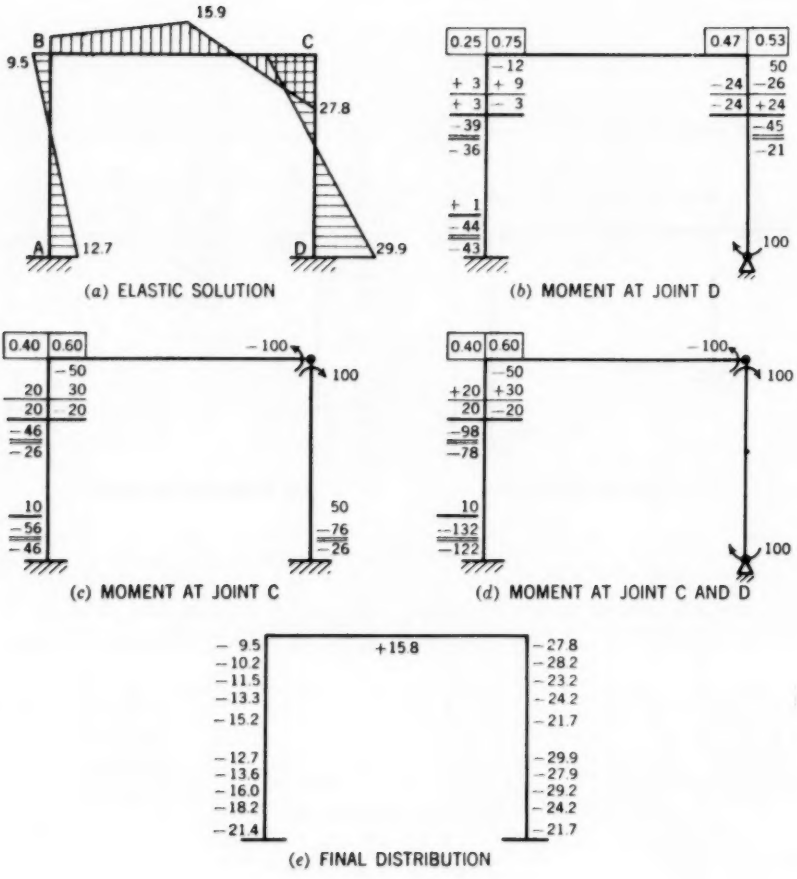


FIG. 8.—ORDER OF DEVELOPMENT OF HINGES FOR EXAMPLE 1